Compressive Multiplexers for Correlated Signals

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***Abstract*—We propose two compressive multiplexers for the efficient sampling of ensembles of correlated signals. We show that we can acquire correlated ensembles, taking advantage of their (a priori-unknown) correlation structure, at a sub-Nyquist rate using simple modulation and filtering architectures. We recast the reconstruction of the ensemble as a low-rank matrix recovery problem from generalized linear measurements. Our theoretical results indicate that we can recover an ensemble of M correlated signals composed of $R \ll M$ independent signals, each bandlimited to $W/2$ Hz, by taking $O(RW \log^q W)$ samples per second, where $q > 1$ is a small constant.**

I. Introduction

In this paper, we propose two compressive multiplexers for the sub-Nyquist acquisition of the ensembles of correlated signals. For each compressive multiplexer, we present a sampling theorem that characterizes the sampling rate sufficient for the successful reconstruction of the signal ensemble. Figure 1 depicts M signals, each of which is bandlimited to $W/2$ radians/sec, that are outputs from a sensor array. Since the signals are bandlimited, they can be captured completely at MW samples per second. In [1], we showed that if the signals are correlated, i.e., M signals can be composed of underlying $R \ll M$ independent signals then using analog preprocessing, we can acquire the ensemble using M analog-to-digital converters (ADCs) with an approximate cumulative sampling rate of $RW \ll MW$. In this paper, we develop sampling architectures that require only one ADC for the efficient acquisition of the ensemble. For this purpose, the conventional time and frequency multiplexing require the single ADC to operate at M times the bandwidth W. We will show that using modulation and filtering, we can acquire the ensemble using a single ADC operating at a rate that roughly scales like R/M times the Nyquist rate.

A. Signal model

We will use notation $\boldsymbol{X_c}(t)$ to denote a signal ensemble of interest and $x_1(t), \dots, x_M(t)$ to denote the individual signals within that ensemble. Conceptually, we may think of $\boldsymbol{X_c}(t)$ as a "matrix" with a finite number M of rows, but with each row containing a bandlimited signal. Our underlying assumption is that the signals in the ensemble are *correlated* in that

$$\boldsymbol{X}_c(t) \approx \boldsymbol{A}\boldsymbol{V}(t), \tag{1}$$

where $\boldsymbol{V}(t)$ is a smaller signal ensemble with R rows, and $\boldsymbol{A}$ is an $M \times R$ matrix with entries $A[m,r]$. We will call $\boldsymbol{A}$

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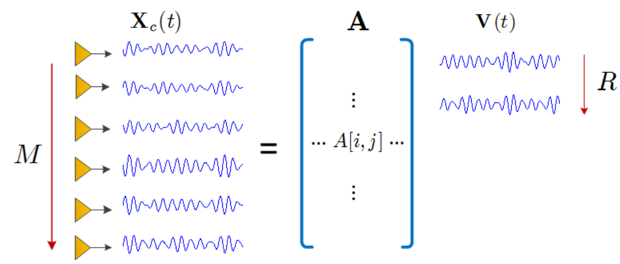

Fig. 1. *Acquire an ensemble of M signals, each bandlimited to B radians per second. The signals are* correlated, *i.e., M signals can be well approximated by the linear combination of R underlying signals. Therefore, we can write M signals in ensemble $\boldsymbol{X}_c(t)$ (on the left) as a tall matrix (a correlation structure) multiplied by an ensemble of R underlying independent signals. Samples $\boldsymbol{X}$ of ensemble $\boldsymbol{X}_c(t)$ inherit the low-rank property. Therefore, the problem of recovering $\boldsymbol{X_c(t)}$ from samples at a sub-Nyquist rate can be recast as a low-rank matrix recovery problem from partial-generalized measurements.*

the *correlation structure* of the ensemble. We can write (1) equivalently as

$$x_m(t) \approx \sum_{r=1}^{R} A[m,r] v_r(t). \tag{2}$$

The only structure we will impose on individual signals is that they are real-valued, bandlimited, and periodic. Thus, the signals live in a finite-dimensional linear subspace, which provides a natural way of discretizing the problem: since what exists in $\boldsymbol{X_c}(t)$ for $t \in [0,1]$ is all there is to know, and each signal can be captured exactly with W equally spaced samples, we can treat the problem as recovering a $M \times W$ matrix. The results can be extended [2] to more general signal models in which the (non-periodic) signal is windowed in time and overlapping sections are reconstructed jointly.

Each bandlimited, periodic signal in the ensemble can be written as

$$x_m(t) = \sum_{f=-B}^{B} \alpha_m[f]\, e^{j2\pi f t},$$

where the Fourier coefficients $\alpha_m[f]$ are complex but have symmetry $\alpha_m[-f] = \alpha_m[f]^*$ to ensure that $x_m(t)$ is real. We can capture $x_m(t)$ perfectly by taking $W = 2B+1$ equally spaced samples per signal. We will call this the $M \times W$ matrix of samples $\boldsymbol{X}$; of course, knowing every entry in this matrix is the same as knowing the entire signal ensemble. We can write

$$\boldsymbol{X} = \boldsymbol{C}\boldsymbol{F},$$

where $\boldsymbol{F}$ is a $W \times W$ normalized discrete Fourier matrix with entries

$$F[\omega, n] = \frac{1}{\sqrt{W}} e^{-j2\pi\omega n/W}, \quad 0 \le \omega, n \le W-1,$$

and $\boldsymbol{C}$ is an $M \times W$ matrix whose rows contain Fourier series coefficients for the signals in $\boldsymbol{X}_c(t)$,

$$C[m, \omega] = \begin{cases} \alpha_m[\omega] & \omega = 0, 1, \ldots, (W-1)/2 \\ \alpha_m[\omega - W]^* & \omega = (W+1)/2, \ldots, W-1 \end{cases}.$$

he matrix $\boldsymbol{F}$ is orthonormal, while $\boldsymbol{C}$ inherits the correlation structure of the original ensemble, since $\boldsymbol{C}$ is isometric with $\boldsymbol{X}$, we will only be concerned with the recovery of $\boldsymbol{C}$.

If the correlation structure $\boldsymbol{A}$ is known then an optimal sampling strategy would be to reduce M signals to R independent signals by an analog vector-matrix multiplier and multiplex the signals using traditional frequency or time multiplexer with the ADC operating at RW samples per second. This sampling rate is optimal as there are only R independent signals in the ensemble. In the rest of this paper, we will assume that the correlation structure $\boldsymbol{A}$ is unknown. We will consider the case in which $\boldsymbol{C}$ is exactly rank R and the case in which $\boldsymbol{C}$ is technically full rank but can be very closely approximated by a low-rank matrix (i.e., the spectrum of singular values decays rapidly).

We recast the problem of signal reconstruction from the sub-Nyquist samples as a low-rank recovery problem from under determined set of linear measurements. The conditions under which $\boldsymbol{C}$ can be recovered from limited measurements have undergone an intensive study in the recent literature [3]–[7]. The sampling architectures, namely, the modulating multiplexer (M-MUX) and the prefiltering-modulating multiplexer (FM-MUX) proposed in this paper produce a series of measurements that are composed of linear combinations of the entries of $\boldsymbol{C}$. As we will show later that in both cases, the recovery of the signal ensemble from the multiplexer output is a low-rank matrix recovery (LRMR) problem from structured random measurements. Such measurement schemes have not been analyzed in the LRMR literature. Our results demonstrate that using structured random sensing, we can successfully recover the low-rank matrix with the number of observations that are within log factors of the degrees of freedom in the low-rank matrix.

Figure 3 shows the M-MUX for the acquisition of correlated signals. In this architecture, each of the signals is randomly modulated independently with a binary waveform alternating at rate $\Omega > W$. Afterwards, the signals are added and sampled uniformly at rate Ω. The random modulation disperses the frequency spectrum of the signals over a wider band Ω that allows us to unmix the signals from the limited samples acquired. Theorem 1 shows that under some mild time dispersion assumptions on the input signals, the M-MUX can acquire the correlated ensemble at a rate that is within log factors of the optimal sampling rate.

Figure 4 depicts the FM-MUX. This architecture is constructed by including a random linear time-invariant (LTI) filter in each channel of the M-MUX. The random filter in each channel disperses the signal energy across time by convolving with a long and diverse impulse response. This ensures that the samples taken by the ADC are almost always non-zero. As a result, the FM-MUX performs equally well for any correlated ensemble regardless of its initial energy distribution. In Theorem 2, we characterize the sampling rate Ω required for the stable reconstruction of the correlated ensemble. The sampling rate will be within log factors of the optimal sampling rate of RW samples per second. To reconstruct the signals from the samples in the M-MUX and the FM-MUX, we solve a nuclear-norm minimization program to recover $\boldsymbol{C}$.

The rest of the paper is organized as follows. Section II is devoted to applications. In Section III, we discuss the nuclear-norm minimization and the conditions on the measurement framework that guarantee the success of the LRMR problem. In Section IV and V, we present the compressive multiplexers and the related sampling theorems.

II. Applications: Micro sensor arrays

Correlated signals arise in recordings form micro sensor arrays used in several applications in biology and robotics, for example, in neuronal recordings from brain tissues in response to visual stimuli. These neuronal recordings are useful to the understanding of neural processes such as information encoding, processing, the classification of cell types, and determining the exact locations of neurons. To record the

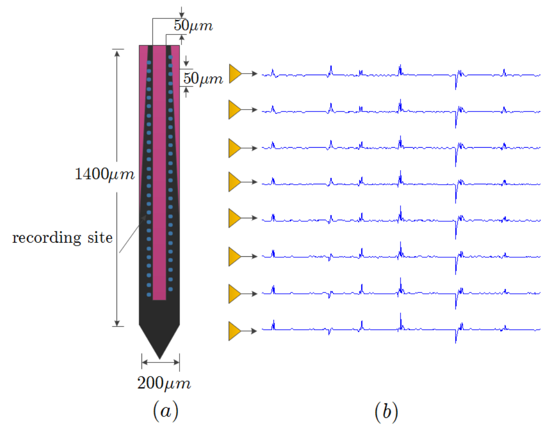

Fig. 2. *a) Photomicrograph of a Polytrode. The shank has planar recording sites spanned 50 μm apart in two columns. Polytrodes with denser site spacing provide highly detailed field recordings and span 1mm. [8] b) Signals recorded by sensors in a real experiment. The signals are closely correlated. The data is taken from the website http://crcns.org.*

neuronal activity in the cortical columns of the brain, a tiny silicon electrode array containing many closely located recording sites is inserted in the brain of a specie (usually a cat or a monkey). Figure 2 shows that the multi-neuron-evoked responses, the signals are highly correlated. These signals are recorded by electrode arrays in a real experiment conducted to monitor the neuronal activity resulting from a visual stimulus. Furthermore, Figure 2 shows an electrode array (on the left), called polytrode, which contains 54 recording sites and a cortical layer of brain (on the right) is also shown. An electrode array may contain tens to hundreds of recording sites. That is, we are recording as many correlated signals, which are multiplexed, continuously sampled by ADCs, and streamed to memory at high quantization resolution. Because the signals are highly correlated and there may be hundreds of such signals in a batch of recordings, the data generated over the course of a typical experiment lasting over 24 hours may reach terabytes. In particular, [9] describes data acquisition hardware for an electrode array containing 512-recording sites. The 512 signals recorded are bandpass filtered, amplified, and then multiplexed using 64:1 analog multiplexers on eight channels. Each of the eight multiplexed signals is sampled at 1.28 MH, which corresponds to a sampling rate of 20 KHz for each signal recorded at the electrode. All the samples are then digitized at 15 MB/s. It is clear that the increasing number of recording sites on an electrode not only increases the sampling burden on the ADCs, but it also increases the data generated in an experiment. This is especially true for experiments lasting several hours. Therefore, it will be useful to acquire data efficiently by taking into account the correlated signal structure that exists in the ensemble. We can deploy the sampling architectures presented in this paper and hence, compressively acquire the signal ensembles not only to effectively use sampling resources but also to minimize the amount of data generated over the course of an experiment.

III. Background

The matrix recovery framework can be used to reproduce the Shannon-Nyquist performance for correlated signals using limited number of samples. Over a time window of fixed length, we obtain the measurements

$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{C}) + \boldsymbol{n},$$

where $\boldsymbol{y} \in \mathbb{R}^{\Omega}$ is the vector of measurements, $\boldsymbol{n} \in \mathbb{R}^{\Omega}$ is the noise vector such that $\|\boldsymbol{n}\|_2 \leq \delta$, and $\mathcal{A}$ is the linear sampling operator. To solve for $\boldsymbol{C}$ in the noiseless case ($\delta = 0$), we use the nuclear-norm minimization program subject to affine constraints as below:

$$\begin{aligned} &\text{minimize} \quad \|\boldsymbol{Z}\|_* \\ &\text{subject to} \quad \boldsymbol{y} = \mathcal{A}(\boldsymbol{Z}). \end{aligned} \tag{3}$$

In the noisy case ($\delta > 0$), we solve the nuclear-norm minimization subject to quadratic constraints:

$$\begin{aligned} &\text{minimize} \quad \|\boldsymbol{Z}\|_* \\ &\text{subject to} \quad \|\boldsymbol{y} - \mathcal{A}(\boldsymbol{Z})\|_2 \leq \delta. \end{aligned} \tag{4}$$

In the following subsections, we provide conditions on the linear map $\mathcal{A}$ required to ensure the stable recovery of the samples of the signal ensemble using the convex optimization program (4). We will begin by a brief description of the dual certificate approach to guarantee the exact recovery using the nuclear-norm minimization.

A. Duality

Suppose $\boldsymbol{C}_0$ is a rank-R matrix with a singular value decomposition given by

$$\boldsymbol{C}_0 = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*. \tag{5}$$

Let $\boldsymbol{u}_k$, and $\boldsymbol{v}_k$ denote the kth column of $\boldsymbol{U}$, and $\boldsymbol{V}$, respectively. We use standard notation for the orthogonal decomposition of $\mathbb{C}^{M\times W} = T \oplus T^{\perp}$, where T is the space formed by elements of the form $\boldsymbol{u}_k\boldsymbol{x}^*$ and $\boldsymbol{y}\boldsymbol{v}_k^*$ for arbitrary $\boldsymbol{x}$ and $\boldsymbol{y}$. The orthogonal complement $T^{\perp}$ is formed by the elements of the form $\boldsymbol{x}\boldsymbol{y}^*$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal to $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ for all $1 \leq k \leq R$. $\mathcal{P}_T$ and $\mathcal{P}_T^{\perp}$ will be used to denote the orthogonal projections on the space T and $T^{\perp}$, respectively.

Let $\partial\|\boldsymbol{C}_0\|_*$ denote the sub differential of the nuclear norm at $\boldsymbol{C}_0$. To exhibit exact recovery, it is enough to show that there exists a matrix $\boldsymbol{Y} \in \partial\|\boldsymbol{C}_0\|_*$ such that the following condition of inexact duality is satisfied. This condition is sufficient for the uniqueness of the minimizer [4], [7].

Lemma 1. *Consider a rank-R matrix $\boldsymbol{C}_0$ as in* (5) *which is feasible for the nuclear-norm minimization* (3). *Assume that*

$$\|\mathcal{P}_T\mathcal{A}^*\mathcal{A}\mathcal{P}_T - \mathcal{P}_T\| \leq \frac{1}{2} \qquad \textit{and} \qquad \|\mathcal{A}\| \leq \gamma.$$

Suppose there exists $\boldsymbol{Y} \in \mathit{Range}(\mathcal{A}^*)$ such that

$$\|\boldsymbol{U}\boldsymbol{V}^* - \mathcal{P}_T(\boldsymbol{Y})\|_F \leq \frac{1}{4\sqrt{2}\gamma} \qquad \textit{and} \qquad \|\mathcal{P}_T^{\perp}(\boldsymbol{Y})\| < \frac{3}{4}$$

Then $\boldsymbol{C}_0$ is the unique solution to the nuclear-norm minimization (3).

Existence of dual certificate above roughly guarantees that a sub gradient of the nuclear norm at the point $\boldsymbol{C}_0$ is perpendicular to the affine feasible set in (3), which is a sufficient condition for the uniqueness of the minimizer.

B. The restricted-isometry property for low-rank matrices

The RIP property for the low-rank matrices [7] can be thought of as an extension of the RIP for sparse vectors. The linear map $\mathcal{A}$ is said to satisfy the R-restricted isometry property if for every integer $1 \leq R \leq \min\{M, W\}$, we have a smallest constant δ_R s.t.

$$(1 - \delta_R)\|\boldsymbol{C}\|_F \leq \|\mathcal{A}(\boldsymbol{C})\|_2 \leq (1 + \delta_R)\|\boldsymbol{C}\|_F$$

for all matrices of rank$(\boldsymbol{C}) \leq R$. Assume that $\delta_{5R} < 0.1$, then the solution $\tilde{\boldsymbol{C}}$ to the optimization program (4) obeys [6]

$$\|\tilde{\boldsymbol{C}} - \boldsymbol{C}\|_F \leq c_0 \cdot \frac{\|\boldsymbol{C} - \boldsymbol{C}_R\|_*}{\sqrt{R}} + c_1\delta,$$

where c_0 and c_1 are small constants that depend on isometry constant δ_{5R}. $\boldsymbol{C}_R$ is the best rank-R approximation of $\boldsymbol{C}$. The solution $\tilde{\boldsymbol{C}}$ is exactly equal to $\boldsymbol{C}$ when the noise strength $\delta = 0$ and rank$(\boldsymbol{C}) \leq R$.

IV. M-MUX: A Compressive Multiplexer for Time-Dispersed Correlated Signals

This section is devoted to the working, mathematical setup, and sampling theorem for the modulating multiplexer (M-MUX) shown in Figure 3. The modulators take input signals $x_1(t), \cdots, x_M(t)$ and multiply them with fixed and known waveforms $d_1(t), \cdots, d_M(t)$, respectively. Each of the $d_k(t)$ is an independent and random binary ± 1 waveform that is constant over a time interval of length $1/\Omega$ (meaning that

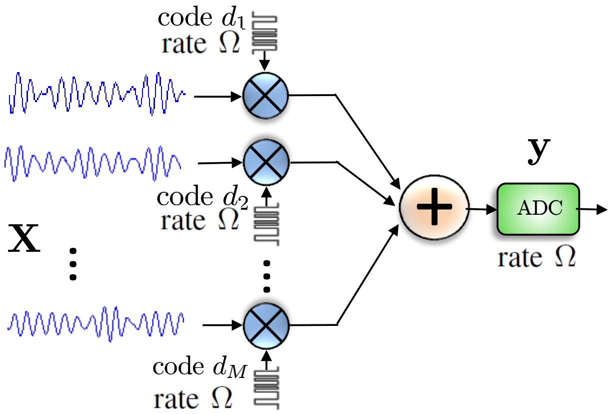

Fig. 3. *The M-MUX for time-dispersed correlated ensembles.*

the switches in the binary waveform occur at a rate greater than the Nyquist sampling rate). The ADC takes Ω samples of $\sum_{k=1}^{M} x_k(t)d_k(t)$ on $[0,1]$. We can write the vector of samples $\boldsymbol{y}$ as

$$\boldsymbol{y} = [\boldsymbol{D}_1\tilde{\boldsymbol{F}}^*, \boldsymbol{D}_2\tilde{\boldsymbol{F}}^*, \cdots, \boldsymbol{D}_M\tilde{\boldsymbol{F}}^*] \cdot \text{vec}(\boldsymbol{C}^*), \qquad (6)$$

where $\boldsymbol{C}$ is the $M \times W$ matrix containing as its rows the Fourier coefficients of the rows of $\boldsymbol{X}_c(t)$, vec(.) takes a matrix and returns a vector obtained by stacking its columns, $\tilde{\boldsymbol{F}}$ is the $W \times \Omega$ matrix consisting of the first W rows of the normalized $\Omega \times \Omega$ DFT matrix, and each of the $\boldsymbol{D}_k$ is an $\Omega \times \Omega$ random diagonal matrix with entries that are samples $\boldsymbol{d}_k \in \mathbb{R}^\Omega$ of $d_k(t)$. Also, $d_k[n] = \pm 1$ with equal probability and the $d_k[n]$ are independent.

Conceptually, the modulator embeds $x(t)$ into a higher dimensional space — this allows us to add several such embedded signals together and then "untangle" them using their structure. Correlated signals that are sparse across time are not efficiently captured by the M-MUX because for such signals the ADC is most of the time sampling zeros, which do not provide any global information about the ensemble. Hence, the M-MUX is more effective for correlated signals that are dispersed across time. This fact is also supported by Theorem 1.

The measurements $\boldsymbol{y} \in \mathbb{R}^\Omega$ in (6) can equivalently be expressed as a linear transformation $\mathcal{A} : \mathbb{C}^{M\times W} \to \mathbb{R}^\Omega$ of matrix $\boldsymbol{C}$ as

$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{C}). \qquad (7)$$

We want to recover low-rank $\boldsymbol{C}$ from a small number of measurements Ω. For this purpose, we use the dual certificate to establish the exact recovery of $\boldsymbol{C}$ from limited measurements; see Section III-A. Let $\boldsymbol{C} = \boldsymbol{U\Sigma V}^*$ be the reduced form SVD of rank-R matrix $\boldsymbol{C}$. We will use the coherence parameter $\mu(V)$ to quantify the degree to which the signals are dispersed across time:

$$\mu(V) = \frac{\Omega}{R} \max_{1\le\omega\le\Omega} \|\boldsymbol{V}^*\boldsymbol{f}_\omega\|_2^2, \qquad (8)$$

where $\boldsymbol{f}_\omega$ is the ωth column of the Fourier matrix $\tilde{\boldsymbol{F}}$. It can be shown that $1 \le \mu(V) \le W/R$, where the lower bound is achieved for signals with energy equally distributed across time.

Theorem 1. *[2] Let $\boldsymbol{C}$ be an $M\times W$ matrix of rank R. Assume that the coherence $\mu(V) \le \mu_0$. Suppose Ω measurements of $\boldsymbol{C}$ are taken through the M-MUX setup* (7)*. If*

$$\Omega \ge C\,(\mu_0 M + W)\, R\beta \log^3(WN)$$

for some $\beta > 1$, then the minimizer $\tilde{\boldsymbol{C}}$ to the problem (3) *is unique and equal to $\boldsymbol{C}$ with probability at least $1 - O(W+N)^{1-\beta}$.*

The sampling theorem above indicates that we can recover the correlated ensemble at a sampling rate close (to with in log factors and a small constant) to the optimal sampling rate RW. This is an improvement over the Nyquist rate by a factor of R/M assuming without loss of generality that $W \ge M$. The above result can also be viewed as a low-rank matrix recovery result using a linear transformation $\mathcal{A}$, which can be applied more efficiently compared to the dense completely random linear operators that are shown to work for matrix recovery. The proof of the result is provided in [2].

The M-MUX has been proposed previously in the literature [10] for the sub-Nyquist acquisition of sparse signals. The principal finding is that if the Fourier spectrum of the input signals can be approximated by $S \ll MW$ active frequency components then [11] shows that the ADC is required to operate at rate $\Omega \approx S \log^q MW$, where $q > 1$ is a small constant. A simple implementation of the M-MUX using a passive averager is discussed in [10].

V. FM-MUX: A Uniform Compressive Multiplexer for Correlated Signals

As mentioned in the previous section, the M-MUX is more effective for efficiently acquiring correlated signals that are dispersed across time. In this section, we present the FM-MUX that is equally effective for any ensemble of correlated signals regardless of its initial energy distribution. To achieve this, we add linear time-invariant (LTI) filters in each channel that force the signal energy to be equally distributed. The FM-MUX is depicted in Figure 4.

The analog preprocessing in the FM-MUX involves modulation by random binary waveforms alternating at rate $\Omega > W$, prefiltering by random LTI filters of bandwidth Ω, and addition over channels. The resultant signal is sampled by a uniform ADC at rate Ω.

As before, the modulators in the FM-MUX take the input signals $x_1(t), \cdots, x_M(t)$ and multiply them with $d_1(t), \cdots, d_M(t)$, respectively. Each of the $d_k(t)$ is an independent and random binary ± 1 waveform that is constant over a time interval of length $1/\Omega$, where $\Omega > W$ and W is the bandwidth of the signals. The LTI filter in the kth channel takes the resultant signal $x_k(t)d_k(t)$, which is bandlimited to $\Omega/2$, and convolves it with a fixed and known impulse response $h_k(t)$. We will assume that we have complete control over $h_k(t)$, even though this brushes aside admittedly important implementation questions. We write the action of the LTI filter $h_k(t)$ as an $\Omega \times \Omega$ circular matrix $\boldsymbol{H}_k$ operating

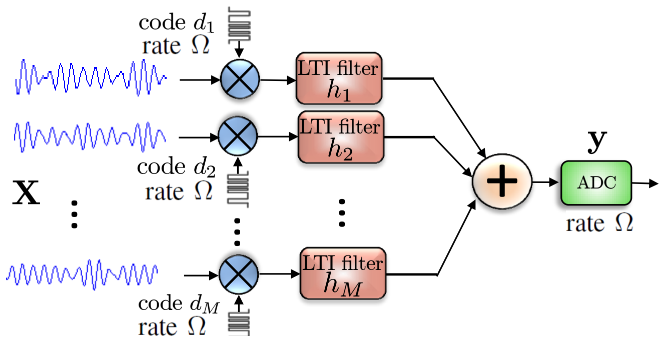

Fig. 4. *The FM-MUX for arbitrary correlated ensemble.*

on the Nyquist rate samples of $x_k(t)d_k(t)$ (the first row of $\boldsymbol{H}$ consists of samples $\boldsymbol{h}_k$ of $h_k(t)$). The circulant matrix $\boldsymbol{H}_k$ is diagonalized by the discrete Fourier transform:

$$\boldsymbol{H}_k = \boldsymbol{F}^* \hat{\boldsymbol{H}}_k \boldsymbol{F},$$

where $\hat{\boldsymbol{H}}_k$ is a diagonal matrix whose entries are $\hat{\boldsymbol{h}}_k = \sqrt{\Omega}\boldsymbol{F}\boldsymbol{h}_k$. The vector $\hat{\boldsymbol{h}}_k$ is a scaled version of the non-zero Fourier series coefficients of $h_k(t)$.

To generate the impulse response, we will use a random unit-magnitude sequence in the Fourier domain [12]. In particular, we will take

$$\hat{h}_k(\omega) = \begin{cases} \pm 1, \text{with prob. } 1/2, & \omega = 0 \\ e^{j\theta_\omega}, \text{ where} & 1 \leq \omega \leq (\Omega-1)/2 \\ \ \theta_\omega \sim \text{Uniform}([0, 2\pi]). & \\ \hat{h}_k(\Omega - \omega + 1)^*, & (\Omega+1)/2 \leq \omega \leq \Omega - 1 \end{cases}.$$

These symmetry constraints are imposed so that $\boldsymbol{h}_k$ (and hence, $h_k(t)$) is real-valued. Conceptually, convolution with $h_k(t)$ disperses a signal over time while maintaining fixed energy (note that $\boldsymbol{H}_k$ is an orthonormal matrix).

After the analog preprocessing by a modulator and a filter in each channel, the signals are combined and sampled uniformly at rate Ω. The samples $\boldsymbol{y} \in \mathbb{R}^{\Omega}$ taken by the ADC can be expressed as

$$\boldsymbol{y} = [\boldsymbol{H}_1 \boldsymbol{D}_1 \tilde{\boldsymbol{F}}^*, \boldsymbol{H}_2 \boldsymbol{D}_2 \tilde{\boldsymbol{F}}^*, \cdots, \boldsymbol{H}_M \boldsymbol{D}_M \tilde{\boldsymbol{F}}^*].\text{vec}(\boldsymbol{C}^*), \quad (9)$$

where all of the notation has already been defined. Hence, the samples are obtained by linear combinations of the entries of an $M \times W$ low-rank $\boldsymbol{C}$. The linear transformation is employing randomness in a structured form. The measurements $\boldsymbol{y}$ can equivalently be expressed as a linear transformation denoted by $\mathcal{A} : \mathbb{C}^{M \times W} \to \mathbb{R}^{\Omega}$ as follows:

$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{C}). \quad (10)$$

To show the exact and stable recovery of the ensemble, we will establish the matrix RIP of operator $\mathcal{A}$; for details see, [2].

Theorem 2. *[2] Fix $\delta \in (0,1)$, then for every integer $1 \leq R \leq M$. The linear map $\mathcal{A} : \mathbb{C}^{M \times W} \to \mathbb{R}^{\Omega}$ in (10) satisfies the R-restricted isometry property:*

$$(1-\delta)\|\boldsymbol{X}\|_F \leq \|\mathcal{A}(\boldsymbol{X})\|_2 \leq (1+\delta)\|\boldsymbol{X}\|_F$$

for all the matrices of rank$(\boldsymbol{X}) \leq R$ with a probability at least $1 - \exp(-c(\delta)\Omega/\log^4(\Omega M))$, whenever $\Omega \geq cRW\log^4(\Omega M)$.

The sampling theorem above shows that the ADC can acquire any correlated ensemble regardless of its initial energy distribution by operating at the optimal sampling rate RW (to within a constant and log factors).

We can also swap the filter and modulator in each channel in Figure 4 to obtain an equivalent architecture. The only difference will be that the filters now will have a bandwidth of WHz instead of ΩHz in the FM-MUX.

VI. Summary

We presented two compressive multiplexers for acquirin ensembles of correlated signals where total number of samples we take scale like bandwidth times the rank (to within log factors) instead of sacling like bandwidth times the number of signals.

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